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SUFFICIENT CONDITIONS FOR UNIFORMLY SECOND-ORDER CONVERGENT SCHEMES FOR STIFF INITIAL-VALUE PROBLEMS

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Abstract—We present a convergence analysis for a class of one-step, exponentially fitted, finite difference schemes for stiff initial-value problems. Such schemes, when applied to the numerical integration of the linear scalar problem

$$\epsilon y' + a(x)\, y = f(x), \qquad x \in (0, X),$$

with $y(0)$ given and where $\epsilon > 0$ is a small parameter, give solutions satisfying a uniform (in ϵ) error estimate. In this paper, a set of sufficient conditions for uniform second-order convergence is derived. The findings differ from those reported in [1–3], and are complemented by the results of numerical experiments which illustrate the effectiveness of the proposed approach.

1. INTRODUCTION

We examine a class of difference schemes for the numerical solution of the stiff initial-value problem

$$\epsilon y' + a(x)\, y = f(x), \qquad x \in (0, X), \tag{1}$$

with $y(0)$ given, where $\epsilon > 0$ is a small parameter, and it is assumed that $a(x)$ and $f(x)$ are smooth functions with

$$a(x) \geq \underline{a} > 0. \tag{2}$$

We are interested in methods whose solutions y_n satisfy an error estimate of the form

$$\max_n |y(x_n) - y_n| \leq C\, h^p, \tag{3}$$

where $y(x_n)$ is the solution of the continuous problem (1) and C is a constant independent of n, h and ϵ. In this paper, we will examine the application of exponentially fitted, one-step methods (see [2,4,5], for example) to approximate the solution of (1) and, in particular, we will establish sufficient conditions for a class of difference schemes where a uniform error estimate (3) holds with $p = 2$.

1.1. Exponential Fitting

Many readers who are familiar with the (extensive) literature on initial-value problems will associate the term "exponential fitting" with the methods of Liniger and Willoughby [6] and Cash [7], for example, each of which can be described as an *a priori*, optimal once and for all choice of the fitting factor. This contrasts with the exponentially fitted methods of Il'in [4], Doolan *et al.* [4], Carroll [5] and others, where the fitting factor is chosen adaptively as the solution evolves.

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As an example, the forward Euler scheme, when applied to approximate the solution of (1), may be written in the form

$$\frac{1}{\rho}(y_{n+1} - y_n) + a_n\, y_n = f_n, \qquad \rho = \frac{h}{\epsilon},$$

with $y_0 = y(0)$. The corresponding exponentially fitted scheme is written as

$$\frac{\sigma_n}{\rho}(y_{n+1} - y_n) + a_n\, y_n = f_n, \tag{4}$$

where σ_n is a (variable) fitting factor, and is chosen so that the exact solution of the constant coefficient homogeneous problem, corresponding to (1), namely

$$\epsilon y' + a\, y = 0, \tag{5}$$

satisfies exactly the corresponding difference scheme

$$\frac{\sigma}{\rho}(y_{n+1} - y_n) + a\, y_n = 0. \tag{6}$$

Substituting the exact solution of (5), $y(x) = e^{-n\rho a}\, y(0)$, into (6) gives

$$\frac{\sigma}{\rho} = \frac{a}{1 - e^{-\rho a}}. \tag{7}$$

The fitting factor (7) is then rewritten in terms of the variable coefficient problem data, i.e.,

$$\frac{\sigma_n}{\rho} = \frac{a_n}{1 - e^{-\rho a_n}}, \tag{8}$$

and used in (4) to estimate y_{n+1} from the recursion

$$y_{n+1} = e^{-\rho a_n}\, y_n + \left[1 - e^{-\rho a_n}\right] \frac{f_n}{a_n}, \tag{9}$$

and the resulting approximation converges uniformly (with order one) to the solution of the continuous problem (see [2]).

1.2. A Motivating Example

The exact solution of the simple problem

$$\epsilon y' + y = x, \qquad x > 0, \qquad y(0) = y_0, \tag{10}$$

is

$$y(x) = (y_0 + \epsilon)\, e^{-x/\epsilon} + x - \epsilon. \tag{11}$$

For the Backward Euler scheme

$$y_{n+1} = \frac{1}{1+\rho}\, y_n + \frac{\rho}{1+\rho}\,(n+1)\, h, \tag{12}$$

we obtain the error equation at $x = h$, namely

$$|y(h) - y_1| = \left|\left(e^{-\rho} - \frac{1}{1+\rho}\right)(y_0 + \epsilon)\right|,$$

and, since approximately

$$\max_{\rho} \left|e^{-\rho} - \frac{1}{1+\rho}\right| = 0.205,$$

we have the error bound

$$|y(h) - y_1| \le 0.205\,|y_0 + \epsilon|. \tag{13}$$

The exponentially fitted Backward-Euler scheme

$$y_{n+1} = e^{-\rho a_{n+1}}\, y_n + \left[1 - e^{-\rho a_{n+1}}\right] \frac{f_{n+1}}{a_{n+1}}, \tag{14}$$

yields, for this problem, the formula

$$y_{n+1} = e^{-\rho}\, y_n + \left(1 - e^{-\rho}\right)(n+1)\,h,$$

and the error bound

$$|y(h) - y_1| = \left|\frac{e^{-\rho} - 1}{\rho} + e^{-\rho}\right| h \le 0.3\,h. \tag{15}$$

The essential difference between (13) and (15) is that the latter bound is uniform, independent of both ρ and y_0 and the fitted scheme has been shown in [4] to be uniformly convergent of order one for Problem (1).

1.3. Second-Order Schemes

1.4. Introduction

We will consider difference schemes to approximate the solution of (1) having the particular form

$$\frac{\sigma_n}{\rho}\,(y_{n+1} - y_n) + \frac{1}{2}\,(a_n\, y_n + a_{n+1}\, y_{n+1}) = \alpha_n\, f_n + \beta_{n+1}\, f_{n+1}, \tag{16}$$

$$\frac{\sigma_n}{\rho} = \frac{\hat{a}_n e^{-\rho \hat{a}_n}}{1 - e^{-\rho \hat{a}_n}}, \tag{17}$$

which simplifies to the recursion

$$y_{n+1} = e^{-\rho \hat{a}_n}\, y_n + \left[1 - e^{-\rho \hat{a}_n}\right] \frac{\alpha_n\, f_n + \beta_{n+1}\, f_{n+1}}{\hat{a}_n}, \tag{18}$$

where $\rho = h/\epsilon$,

$$\hat{a}_n = \frac{1}{2}\,(a_n + a_{n+1}),$$

and α and β are parameters which depend on a and ρ.

1.4.1. Motivation for Choosing the Coefficients

In the next section, we wish to establish conditions on α and β so that a uniform error estimate (3) holds with $p = 2$. However, in order to determine appropriate expressions for these coefficients, it is instructive to examine the exact solution of (1) when written in the form

$$y(x+h) = e^{-I(x,x+h)}\, y(x) + \frac{1}{\epsilon}\, e^{-I(0,x+h)} \int_x^{x+h} f(s)\, e^{I(0,s)}\, ds, \tag{19}$$

where

$$I\,(c,d) = \frac{1}{\epsilon} \int_c^d a(z)\, dz.$$

With $x = x_n$ and $x_{n+1} = x_n + h$, assume that $f(x)$ is linear in the interval $[x_n, x_{n+1}]$, so that we can write

$$f(s) = f_n + \frac{1}{h}\,(f_{n+1} - f_n)\,(s - x_n).$$

Using the definition of $\hat{a}_n$ in (24), observing that

$$\left| \frac{1}{h} \int_{x_n}^{x_{n+1}} a(z)\, dz - \hat{a}_n \right| \le C h^2,$$

where C is a constant independent of n and h (see [2, Theorem I.9.1.]), and with the assumption of the linearity of f, we obtain from (19) the approximation

$$\begin{aligned} y(x_{n+1}) &\approx e^{-\rho \hat{a}_n} y(x_n) + \frac{1}{\hat{a}_n} \left[\left\{ f_n - \frac{1}{\hat{a}_n} (f_{n+1} - f_n) \right\} \left(1 - e^{-\rho \hat{a}_n}\right) + f_{n+1} - f_n \right] \\ &= e^{-\rho \hat{a}_n} y(x_n) + \left(1 - e^{-\rho \hat{a}_n}\right) \frac{\psi_n f_n + (1 - \psi_n) f_{n+1}}{\hat{a}_n}, \end{aligned} \tag{20}$$

where

$$\psi_n = \frac{1}{\rho\, \hat{a}_n} - \frac{e^{-\rho \hat{a}_n}}{1 - e^{-\rho \hat{a}_n}}. \tag{21}$$

We propose to consider convergence properties of schemes of type (18) with

$$\alpha_n = r_n\, \psi_n, \qquad \beta_n = s_n\, (1 - \psi_n), \tag{22}$$

and derive sufficient conditions for r_n and s_n so that (18) give solutions which are uniformly convergent of order 2.

We remark that such a scheme was proposed in [3,4]. It consists of the choice

$$r_n = \frac{\hat{a}_n}{a_n}, \qquad s_n = \frac{\hat{a}_n}{a_{n+1}}, \tag{23}$$

and was reported in [1–3] to give solutions which are uniformly convergent of order 2.

2. ANALYTICAL TOOLS

2.1. Notation

In what follows, we will use the notation

$$\begin{aligned} &\hat{a}_n = \frac{1}{2}(a_n + a_{n+1}), && \hat{a}_{2n} = \frac{1}{2}\left(a_n + a_{n+\frac{1}{2}}\right), \\ &\hat{a}_{2n+1} = \frac{1}{2}\left(a_{n+\frac{1}{2}} + a_{n+1}\right), && \hat{f}_n = \alpha_n f_n + \beta_n f_{n+1}, \\ &\hat{f}_{2n} = \alpha_{2n} f_n + \beta_{2n} f_{n+1/2}, && \hat{f}_{2n+1} = \alpha_{2n+1} f_{n+1/2} + \beta_{2n+1} f_{n+1}, \\ &\tau_n = e^{-1/2 \rho \hat{a}_n}, && \tau_{2n} = e^{-1/2 \rho \hat{a}_{2n}}, \\ &\tau_{2n+1} = e^{-1/2 \rho \hat{a}_{2n+1}}. \end{aligned} \tag{24}$$

2.2. General Convergence Principle

In order to establish sufficient conditions for uniform convergence, we need some background results. We first state two results, the first of which guarantee that a discrete maximum principle holds and the second leads to a discrete stability result for the solution of the difference Equation (18).

LEMMA 1. *Let v_n be a mesh function and $X > 0$. Suppose $v_0 \ge 0$ and $L^h v_n \ge 0$ for all $0 \le n\,h \le X$. Then, $v_n \ge 0$ for all $0 \le n\,h \le X$.*

PROOF 1. See [2, Lemma 5.1].

Lemma 2. *Let v_n be a mesh function and $X > 0$. Then, for $0 \le n\,h \le X$, we have*

$$|v_n| \le |v_0| + \frac{1}{s} \max_{0 \le nh \le X} |L^h v_n|,$$

for some non-zero constant s.

Proof 2. See [2, Lemma 5.2].

The latter result allows us to deduce that the solution y_n of (18) satisfies

$$|y_n| \le |y_0| + \frac{1}{s} \max_{0 \le nh \le X} |f(x_n)|.$$

We now state a very useful result which is based on the general convergence principle of [2], and which will allow us to derive convergence results using the two-mesh method.

Theorem 1. *Let y be the solution of (1), y_n^h a difference approximation obtained on a uniform mesh of length h and $y_{2n}^{h/2}$ the corresponding approximation on a mesh of length $h/2$. Let C_1 and C_2 be constants independent of h. Then, for all $n \ge 0$, all $0 < h \le h_0$, and all $\epsilon > 0$*

$$|y(n\,h) - y_n^h| \le C_1\,h^p \tag{25}$$

if, and only if,

$$\lim_{h \to 0} |y(n\,h) - y_n^h| = 0 \tag{26}$$

and

$$|y_n^h - y_{2n}^{h/2}| \le C_2\,h^p. \tag{27}$$

Furthermore, C_1 is independent of ϵ if and only if C_2 is.

Proof 3. See [2, Theorem I.5.1].

Lemma 3. *(Solution of a recurrent inequality): Given a sequence of numbers $\{\xi_n\}_{n=0}^{\infty}$, that are known to satisfy inequalities of the form*

$$\xi_{n+1} \le A\,\xi_n + B,$$

where A and B are certain nonnegative constants, independent of n, then

$$|\xi_n| \le A^n |\xi_0| + \frac{A^n - 1}{A - 1} B, \qquad A \ne 1.$$

Proof 4. See [7, p. 20].

3. COVERGENCE ANALYSIS

3.1. Sufficient Conditions

We first establish conditions for the validity of the "double-mesh" condition (27) of Theorem 1 with $p = 2$.

Using the notation of (24), since

$$y_{2n+1}^{h/2} = \tau_{2n}\,y_{2n}^{h/2} + (1 - \tau_{2n})\,\frac{\hat{f}_{2n}}{\hat{a}_{2n}},$$

we obtain

$$\begin{aligned} y_{2n+2}^{h/2} &= \tau_{2n+1}\,y_{2n+1}^{h/2} + (1 - \tau_{2n+1})\,\frac{\hat{f}_{2n+1}}{\hat{a}_{2n+1}} \\ &= \tau_{2n+1} \left[\tau_{2n}\,y_{2n}^{h/2} + (1 - \tau_{2n})\,\frac{\hat{f}_{2n}}{\hat{a}_{2n}} \right] + (1 - \tau_{2n+1})\,\frac{\hat{f}_{2n+1}}{\hat{a}_{2n+1}} \\ &= \tau_{2n}\,\tau_{2n+1}\,y_{2n}^{h/2} + \tau_{2n+1}\,(1 - \tau_{2n})\,\frac{\hat{f}_{2n}}{\hat{a}_{2n}} + (1 - \tau_{2n+1})\,\frac{\hat{f}_{2n+1}}{\hat{a}_{2n+1}}. \end{aligned} \tag{28}$$

On subtracting (18) from (28), we then have

$$\left|y_{2n+2}^{h/2} - y_{n+1}^{h}\right| \le A\left|y_{2n}^{h/2} - y_{n}^{h}\right| + |B_1|\left|y_{2n}^{h/2}\right| + |B_2|,$$

where

$$A = \tau_n^2, \tag{29}$$

$$B_1 = \tau_{2n}\,\tau_{2n+1} - \tau_n^2, \tag{30}$$

$$B_2 = \tau_{2n+1}\,(1-\tau_{2n})\,\frac{\hat{f}_{2n}}{\hat{a}_{2n}} + (1-\tau_{2n+1})\,\frac{\hat{f}_{2n+1}}{\hat{a}_{2n+1}} - \left(1-\tau_n^2\right)\frac{\hat{f}_n}{\hat{a}_n}, \tag{31}$$

and we wish to prove that

$$0 \le |A| \le 1 \tag{32}$$

and

$$|B_1| \le C\,h^2, \tag{33}$$

$$|B_2| \le C\,h^2, \tag{34}$$

where C is a (generic) constant independent of n, h and ϵ. From the definition of $a(x)$ in (2), condition (32) is obvious. The remaining conditions, (33) and (34), will be established using the following results.

LEMMA 4.

$$|B_1| \le Ch^2.$$

PROOF 5. Using the notation of the previous section and (30), it follows that

$$\begin{aligned}
|B_1| &= \left|\tau_{2n}\,\tau_{2n+1} - \tau_n^2\right| \\
&= \tau_n^2\left|1 - e^{-\rho\left(\hat{a}_n - \frac{1}{2}(\hat{a}_{2n} - \hat{a}_{2n+1})\right)}\right| \\
&\le \rho\,\tau_n^2\left|\hat{a}_n - \frac{1}{2}\left(\hat{a}_{2n} - \hat{a}_{2n+1}\right)\right| \\
&\le \left|\hat{a}_n - \frac{1}{2}\left(\hat{a}_{2n} - \hat{a}_{2n+1}\right)\right| \\
&\le C\,h^2,
\end{aligned}$$

where we have used the fact that $\rho\, e^{-\rho \hat{a}_n} \le 1$.

We next examine (31) and look for conditions under which (34) holds. If we assume that $f \in C^3[0, X]$, then Taylor expanding for f about x_n in (31) yields

$$|B_2| \le |B_{21}|\,|f_n| + h|B_{22}|\,|f_n'| + O(h^2),$$

where

$$\begin{aligned}
B_{21} &= \tau_{2n+1}\,(1-\tau_{2n})\,\frac{\alpha_{2n}+\beta_{2n}}{\hat{a}_{2n}} + (1-\tau_{2n+1})\,\frac{\alpha_{2n+1}+\beta_{2n+1}}{\hat{a}_{2n+1}} - \left(1-\tau_n^2\right)\frac{\alpha_n+\beta_n}{\hat{a}_n} \\
&= (1-\tau_{2n+1})\left(\frac{\alpha_{2n+1}+\beta_{2n+1}}{\hat{a}_{2n+1}} - \frac{\alpha_{2n}+\beta_{2n}}{\hat{a}_{2n}}\right) \\
&\quad + (1-\tau_{2n}\tau_{2n+1})\left(\frac{\alpha_{2n}+\beta_{2n}}{\hat{a}_{2n}} - \frac{\alpha_n+\beta_n}{\hat{a}_n}\right) + O(h^2)
\end{aligned} \tag{35}$$

using (33), and

$$B_{22} = (1-\tau_{2n+1})\left(\frac{\frac{1}{2}\,\alpha_{2n+1}+\beta_{2n+1}}{\hat{a}_{2n+1}} - \frac{\beta_n}{\hat{a}_n}\right) + \tau_{2n+1}\,(1-\tau_{2n})\left(\frac{\frac{1}{2}\,\beta_{2n}}{\hat{a}_{2n}} - \frac{\beta_n}{\hat{a}_n}\right). \tag{36}$$

This allows us to replace (34) by two further conditions, namely

$$|B_{21}| \le Ch^2, \tag{37}$$
$$|B_{22}| \le Ch. \tag{38}$$

First, we consider (35). Using (22) and (21), we obtain

$$\begin{aligned}
B_{21} &= (1-\tau_{2n}\,\tau_{2n+1})\,\frac{1}{\rho}\left(\frac{r_{2n}-s_{2n}}{\frac{1}{2}\,\hat{a}_{2n}^2}-\frac{r_n-s_n}{\hat{a}_n^2}\right)\\
&\quad+(1-\tau_{2n+1})\,\frac{1}{\rho}\left(\frac{r_{2n+1}-s_{2n+1}}{\frac{1}{2}\,\hat{a}_{2n+1}^2}-\frac{r_{2n}-s_{2n}}{\frac{1}{2}\,\hat{a}_{2n}^2}\right)\\
&\quad+(1-\tau_{2n}\,\tau_{2n+1})\left(\frac{s_{2n}-r_{2n}\,\tau_{2n+1}}{\hat{a}_{2n}\,(1-\tau_{2n+1})}-\frac{s_n-r_n\,\tau_{2n}\,\tau_{2n+1}}{\hat{a}_n\,(1-\tau_{2n}\,\tau_{2n+1})}\right)\\
&\quad+(1-\tau_{2n+1})\left(\frac{s_{2n+1}-r_{2n+1}\,\tau_{2n+1}}{\hat{a}_{2n+1}\,(1-\tau_{2n+1})}-\frac{s_{2n}-r_{2n}\,\tau_{2n}}{\hat{a}_{2n}\,(1-\tau_{2n})}\right)\\
&=(1-\tau_{2n}\,\tau_{2n+1})\,\frac{1}{\rho}\left(\frac{r_{2n}-s_{2n}}{\frac{1}{2}\,\hat{a}_{2n}^2}-\frac{r_n-s_n}{\hat{a}_n^2}\right)\\
&\quad+(1-\tau_{2n+1})\,\frac{1}{\rho}\left(\frac{r_{2n+1}-s_{2n+1}}{\frac{1}{2}\,\hat{a}_{2n+1}^2}-\frac{r_{2n}-s_{2n}}{\frac{1}{2}\,\hat{a}_{2n}^2}\right)\\
&\quad+\left(\frac{s_{2n+1}}{\hat{a}_{2n+1}}-\frac{s_n}{\hat{a}_n}\right)+\tau_{2n+1}\left(\frac{s_{2n}}{\hat{a}_{2n}}-\frac{r_{2n+1}}{\hat{a}_{2n+1}}\right)+\tau_{2n}\,\tau_{2n+1}\left(\frac{r_n}{\hat{a}_n}-\frac{r_{2n}}{\hat{a}_{2n}}\right)
\end{aligned}$$

so that (33) can be replaced by the five following requirements for r and s:

$$\left|\frac{1}{\rho}\left\{\frac{r_{2n}-s_{2n}}{\frac{1}{2}\,\hat{a}_{2n}^2}-\frac{r_n-s_n}{\hat{a}_n^2}\right\}\right| \le Ch^2, \tag{39}$$

$$\left|\frac{1}{\rho}\left\{\frac{r_{2n+1}-s_{2n+1}}{\frac{1}{2}\,\hat{a}_{2n+1}^2}-\frac{r_{2n}-s_{2n}}{\frac{1}{2}\hat{a}_{2n}^2}\right\}\right| \le Ch^2, \tag{40}$$

$$\left|\frac{s_{2n+1}}{\hat{a}_{2n+1}}-\frac{s_n}{\hat{a}_n}\right| \le Ch^2, \tag{41}$$

$$\left|\frac{s_{2n}}{\hat{a}_{2n}}-\frac{r_{2n+1}}{\hat{a}_{2n+1}}\right| \le Ch^2, \tag{42}$$

$$\left|\frac{r_n}{\hat{a}_n}-\frac{r_{2n}}{\hat{a}_{2n}}\right| \le Ch^2. \tag{43}$$

Next, consider (36). Again, using (22) and (21), we find that

$$\begin{aligned}
\frac{\frac{1}{2}\,\alpha_{2n+1}+\beta_{2n+1}}{\hat{a}_{2n+1}}-\frac{\beta_n}{\hat{a}_n} &= \frac{1}{\rho}\left(\frac{r_{2n+1}-s_{2n+1}}{\hat{a}_{2n+1}^2}\right)+\frac{1}{\rho}\left(\frac{s_n}{\hat{a}_n^2}-\frac{s_{2n+1}}{\hat{a}_{2n+1}^2}\right)\\
&\quad+\frac{1}{\hat{a}_{2n+1}}\left(\frac{s_{2n+1}-\frac{1}{2}\,r_{2n+1}\,\tau_{2n+1}}{1-\tau_{2n+1}}\right)-\frac{1}{\hat{a}_n}\left(\frac{s_n}{1-\tau_{2n}\,\tau_{2n+1}}\right)
\end{aligned}$$

and

$$\frac{\frac{1}{2}\,\beta_{2n}}{\hat{a}_{2n}}-\frac{\beta_n}{\hat{a}_n}=\frac{1}{\rho}\left(\frac{s_n}{\hat{a}_n^2}-\frac{s_{2n}}{\hat{a}_{2n}^2}\right)+\frac{1}{\hat{a}_{2n}}\left(\frac{\frac{1}{2}\,s_{2n}}{1-\tau_{2n}}\right)-\frac{1}{\hat{a}_n}\left(\frac{s_n}{1-\tau_{2n}\,\tau_{2n+1}}\right)$$

and, in addition to conditions (39) to (43), (38) requires the following two estimates to hold:

$$\left|\frac{1}{\rho}\left\{\frac{r_{2n+1}-2s_{2n+1}}{\hat{a}_{2n+1}^2}+\frac{s_n}{\hat{a}_n^2}\right\}\right| \le Ch \tag{44}$$

$$\left|\frac{1}{\rho}\left\{\frac{s_{2n}}{\hat{a}_{2n}^2}-\frac{s_n}{\hat{a}_n^2}\right\}\right| \le Ch \tag{45}$$

We have, therefore, proved the following lemma.

LEMMA 5. *Let y^h denote the solution of the difference scheme (18), (22) and (21), at the point $x_n = n\,h$, using a uniform stepsize h, and let $y^{h/2}$ denote the corresponding solution at x_n using a uniform stepsize $h/2$. If r_n and s_n satisfy conditions (39)–(45), then, for all $n \ge 0$, all $0 < h \le h_0$, and all $\epsilon > 0$*

$$\left|y_n^h - y_{2n}^{h/2}\right| \le Ch^2, \tag{46}$$

where C is a constant independent of n, h and ϵ.

PROOF 6. The seven conditions, (39)–(45), were derived so that $\left|y_{n+1}^h - y_{2n+2}^{h/2}\right| \le Ch^2$. Using Lemma 3, the proof follows immediately.

LEMMA 6. *Let $y(n\,h)$ denote the solution of the continuous problem (1) and let y^h be the difference approximation obtained from (18) at the point $x_n = n\,h$, using a uniform stepsize h, $n = 0, 1, \dots, N$ and $N\,h = X$. Then,*

$$\lim_{h\to 0}\left|y(nh) - y_n^h\right| = 0.$$

THEOREM 2. *Let y be the solution of (1) with $X > 0$ and we assume that $a(x) \ge \underline{a} > 0$ for all $0 \le x \le X$, and let y^h be the solution of (18). Then, for all $n \ge 0$, $n = 0, 1, \dots, N$, $Nh = X$, all $0 < h \le h_0$, and all $\epsilon > 0$*

$$\left|y(nh) - y_n^h\right| \le Ch^2, \tag{47}$$

where C is a constant independent of n, h and ϵ.

PROOF 7. Lemmas 6 and 5 establish the validity of conditions (26) and (27) of Theorem 1. The proof follows from the statement of Theorem 1 with $p = 2$.

3.2. An Application

The difference scheme defined by the coefficients

$$r_n = \frac{\hat{a}_n}{a_n}, \qquad s_n = \frac{\hat{a}_n}{a_{n+1}},$$

that is,

$$y_{n+1} = e^{-\rho\hat{a}_n} y_n + \left[1 - e^{-\rho\hat{a}_n}\right]\frac{\alpha_n f_n + \beta_{n+1} f_{n+1}}{\hat{a}_n},$$

$$\alpha_n = \frac{\hat{a}_n}{a_n}\left[\frac{1}{\rho\hat{a}_n} - \frac{e^{-\rho\hat{a}_n}}{1 - e^{-\rho\hat{a}_n}}\right], \quad \beta_n = \frac{\hat{a}_n}{a_{n+1}}\left[\frac{1}{1 - e^{-\rho\hat{a}_n}} - \frac{1}{\rho\hat{a}_n}\right]$$

was reported in [1–3] to give solutions which are uniformly convergent of order 2. From the previous section, however, we find using (35) that

$$\begin{aligned}
B_{21} &= (1 - \tau_{2n}\tau_{2n+1})\frac{1}{\rho}\left(\frac{r_{2n} - s_{2n}}{\frac{1}{2}\hat{a}_{2n}^2} - \frac{r_n - s_n}{\hat{a}_n^2}\right) \\
&\quad + (1 - \tau_{2n+1})\frac{1}{\rho}\left(\frac{r_{2n+1} - s_{2n+1}}{\frac{1}{2}\hat{a}_{2n+1}^2} - \frac{r_{2n} - s_{2n}}{\frac{1}{2}\hat{a}_{2n}^2}\right) \\
&= \frac{1 - \tau_{2n}\tau_{2n+1}}{\rho}\left\{\frac{2}{\hat{a}_{2n}}\left(\frac{1}{a_n} - \frac{1}{a_{n+\frac{1}{2}}}\right) - \frac{1}{\hat{a}_n}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)\right\} \\
&\quad + \frac{1 - \tau_{2n+1}}{\rho}\left\{\frac{2}{\hat{a}_{2n+1}}\left(\frac{1}{a_{n+\frac{1}{2}}} - \frac{1}{a_{n+1}}\right) - \frac{2}{\hat{a}_{2n}}\left(\frac{1}{a_n} - \frac{1}{a_{n+\frac{1}{2}}}\right)\right\} \\
&\approx \frac{1}{\rho}\left\{Ch^2\right\} \\
&= O(h\epsilon),
\end{aligned}$$

which suggests that the scheme is not uniformly second-order accurate. This is reinforced by an experimental determination of the order of convergence in Section 5.

4. NUMERICAL RESULTS

We consider the application of various difference schemes to approximate the solutions of a number of test problems. The difference schemes include

S1 The Trapezoidal Rule
S2 The exponentially fitted Trapezoidal Rule [2]
S3 Linear interpolation scheme (linear-psi)
S4 The scheme of Doolan & Schilders [1,2]
S5 The scheme of Doolan & Schilders [1,2] with α_n and β_n unchanged but with

$$\hat{a}_n = \frac{1}{6}\left(a_n + 4a_{n+\frac{1}{2}} + a_{n+1}\right)$$

elsewhere.

The problems we consider are

P1 Constant $a(x)$, linear $f(x)$:

$$\epsilon\, y' + y = x, \qquad x \in [0,1],$$

with $y(0) = 1$ and $\epsilon = 10^{-2}$.

P2 Linear $a(x)$, linear $f(x)$:

$$\epsilon\, y' + (1+x)\, y = 1 + x, \qquad x \in [0,1],$$

with $y(0) = 2$ and $\epsilon = 1$.

P3 Constant $a(x)$, quadratic $f(x)$:

$$\epsilon\, y' + y = x^2, \qquad x \in [0,1],$$

with $y(0) = 1$ and $\epsilon = 10^{-2}$.

P4 Constant $a(x)$, nonlinear $f(x)$:

$$\epsilon\, y' + y = g(x) + \epsilon\, g'(x), \qquad x \in [0,1],$$

with $y(0) = 10$, $\epsilon = 5 \times 10^{-3}$ and

$$g(x) = 10 - (10 + x)\, e^{-x}.$$

P5 Nonlinear $a(x)$, constant $f(x)$:

$$\epsilon\, y' + \frac{1}{1+x}\, y = 1, \qquad x \in [0,1],$$

with $y(0) = 1$ and $\epsilon = 10^{-2}$.

For each of the problems, we measured the maximum error on $[0,1]$ for a range of (uniform) integration stepsizes h and the results are summarised in the tables below. Note that schemes S3, S4 and S5 are exact for problem P1 ($a(x)$ is constant and $f(x)$ is linear), while S2, S4 and S5 are exact for problem P2 (both $a(x)$ and $f(x)$ are linear).

Table 1. Results for Problem 1.

h	S1	S2	S3, S4, S5
$\frac{1}{16}$	.52E+0	.21E-1	exact
$\frac{1}{32}$	.27E+0	.71E-2	
$\frac{1}{64}$	.88E-1	.20E-2	
$\frac{1}{128}$	.20E-1	.50E-3	
$\frac{1}{256}$	.47E-2	.13E-3	
$\frac{1}{512}$	.12E-2	.32E-4	

Table 2. Results for Problem 2.

h	S1	S3	S2, S4, S5
$\frac{1}{16}$	.15E-3	.25E-3	exact
$\frac{1}{32}$	.37E-4	.63E-4	
$\frac{1}{64}$	.94E-5	.16E-4	
$\frac{1}{128}$	.23E-5	.40E-5	
$\frac{1}{256}$	.59E-6	.99E-6	
$\frac{1}{512}$	.15E-6	.25E-6	

Table 3. Results for Problem 3.

h	S1	S2	S3, S4, S5
$\frac{1}{16}$	.52E+0	.41E-1	.43E-3
$\frac{1}{32}$	.26E+0	.14E-1	.14E-3
$\frac{1}{64}$	.87E-1	.38E-2	.39E-4
$\frac{1}{128}$	.20E-1	.99E-3	.10E-4
$\frac{1}{256}$	.47E-2	.25E-3	.25E-5
$\frac{1}{512}$	.12E-2	.62E-4	.64E-6

Table 4. Results for Problem 4.

h	S1	S2	S3, S4, S5
$\frac{1}{16}$	.72E+1	.23E+0	.10E-2
$\frac{1}{32}$	.52E+1	.95E-1	.42E-3
$\frac{1}{64}$	.26E+1	.31E-1	.14E-3
$\frac{1}{128}$	.87E+0	.86E-2	.38E-4
$\frac{1}{256}$	.20E+0	.22E-2	.98E-5
$\frac{1}{512}$	.47E-1	.56E-3	.25E-5

Table 5. Results for Problem 5.

h	S1	S3	S2	S4	S5
$\frac{1}{16}$	.54E-2	.22E-1	.22E-1	.22E-3	.86E-3
$\frac{1}{32}$	.27E-2	.70E-2	.70E-2	.70E-4	.23E-3
$\frac{1}{64}$	.88E-3	.19E-2	.19E-2	.19E-4	.58E-4
$\frac{1}{128}$	.20E-3	.49E-3	.49E-3	.49E-5	.15E-5
$\frac{1}{256}$	.47E-4	.12E-3	.12E-3	.12E-5	.37E-5
$\frac{1}{512}$	.12E-4	.31E-4	.31E-4	.31E-6	.92E-6

5. EXPERIMENTAL DETERMINATION OF THE ORDER OF UNIFORM CONVERGENCE

The test is based on the General Convergence Principle. Following [2], we define the quantities

$$z_{k,\epsilon} = \max_n \left| y_n^{h/2^k} - y_{2n}^{h/2^{k+1}} \right|, \qquad k = 0, 1, \dots,$$

where the maximum is taken over all mesh points of the coarser of the two meshes and h is the mesh width of the coarsest mesh on which we solve the problem. Clearly, $z_{k,\epsilon}$ is the maximum difference between the solutions on two successive meshes.

For convergent solutions of any difference scheme, the condition

$$z_{k,\epsilon} \le C \left(\frac{h}{2}\right)^p, \qquad k = 0, 1, \dots,$$

suffices to prove uniform convergence of order p, provided that C and p are independent of k and ϵ.

For an experimental determination of the order of uniform convergence, we first compute a set of values of $z_{k,\epsilon}$ for various values of ϵ. An approximate value for p is then determined from the quantities $z_{k,\epsilon}$ in the following way. Assuming that

$$z_{k,\epsilon} \le C_\epsilon \left(\frac{h}{2}\right)^{p_\epsilon}, \qquad k \ge 0,$$

where C_ϵ and p_ϵ are independent of k, we are motivated to put

$$p_{k,\epsilon} = \log_2\left(\frac{z_{k,\epsilon}}{z_{k+1,\epsilon}}\right), \qquad k = 0, 1, \ldots.$$

This gives a set of estimates of the classical rate of convergence associated with the problem corresponding to this value of ϵ. We combine these into a single estimate by taking the average, which we denote by $\overline{p}_\epsilon$. Then our final estimate for the rate of uniform convergence is taken to be the minimum value of $\overline{p}_\epsilon$ over all ϵ considered.

In our experiments, we have chosen

$$h = \frac{1}{2^{k+3}}, \qquad k = 0, 1, \ldots, 4,$$

and

$$\epsilon = \frac{1}{2^{k+1}}, \qquad k = 0, 1, \ldots, 8.$$

For the five problems given, we compute an estimate of the experimental order of convergence and the results are as shown in the table below.

The results show that, for the general problems P3, P4 and P5, where one of $f(x)$ or $a(x)$ is neither a constant nor a linear function of x, scheme S4 of [1,2] is not second-order uniformly convergent. The conditions established earlier illustrated that the rate of convergence was asymptotically $O(h\epsilon)$.

Table 6. Experimental order of uniform convergence.

Problem	S1	S2	S3	S4	S5
P1	1.17	1.43	2.00	2.00	2.00
P2	1.15	2.00	1.42	2.00	2.00
P3	1.17	1.41	1.43	1.43	1.43
P4	1.17	1.43	1.42	1.42	1.42
P5	1.19	1.45	1.45	1.45	1.93

6. CONCLUSIONS

Within the framework of the General Convergence Principle, sufficient conditions were established for uniform convergence for a class of one-step, exponentially fitted, finite difference schemes when applied to approximate the solutions of stiff, linear initial-value problems. Although the analysis was applied directly to the scheme of Doolan and Schilders [1], the method of proof is applicable to more general schemes of arbitrary order. The analytical results were complemented by numerical experiments which illustrated a practical way to estimate orders of (uniform) convergence.

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